

Stability of transition waves and positive entire solutions of Fisher-KPP equations with time and space dependence

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Abstract. This paper is concerned with the stability of transition waves and strictly positive entire solutions of random and nonlocal dispersal evolution equations of Fisher-KPP type with general time and space dependence, including time and space periodic or almost periodic dependence as special cases. We first show the existence, uniqueness, and stability of strictly positive entire solutions of such equations. Next, we show the stability of uniformly continuous transition waves connecting the unique strictly positive entire solution and the trivial solution zero and satisfying certain decay property at the end close to the trivial solution zero (if it exists). The existence of transition waves has been studied in [34, 39, 45, 46, 61] for random dispersal Fisher-KPP equations with time and space periodic dependence, in [41, 42, 43, 51, 52, 53, 58, 63] for random dispersal Fisher-KPP equations with quite general time and/or space dependence, and in [17, 48, 56] for nonlocal dispersal Fisher-KPP equations with time and/or space periodic dependence. The stability result established in this paper implies that the transition waves obtained in many of the above mentioned papers are asymptotically stable for well-fitted perturbation. Up to the author's knowledge, it is the first time that the stability of transition waves of Fisher-KPP equations with general time and space dependence is studied.

Key words. Fisher-KPP equation, random dispersal, nonlocal dispersal, transition wave, positive entire solution, stability, almost periodic.

Mathematics subject classification. 35B08, 35C07, 35K57, 45J05, 47J35, 58D25, 92D25.

1 Introduction

The current paper is devoted to the study of the stability of transition waves and entire positive solutions of dispersal evolution equations of the form,

$$\frac{\partial u}{\partial t} = \mathcal{A}u + uf(t, x, u), \quad x \in \mathbb{R}, \quad (1.1)$$

where $\mathcal{A}u(t, x) = u_{xx}(t, x)$ or $\mathcal{A}u(t, x) = \int_{\mathbb{R}} \kappa(y - x)u(t, y)dy - u(t, x)$ for some nonnegative smooth function $\kappa(\cdot)$ with $\kappa(z) > 0$ for $\|z\| < r_0$ and some $r_0 > 0$, $\kappa(z) = 0$ for $\|z\| \geq r_0$,

and $\int_{\mathbb{R}} \kappa(y) dy = 1$, and $f(t, x, u)$ is of Fisher-KPP type in u . More precisely, we assume that $f(t, x, u)$ satisfies the following standing assumption.

(H0) $f(t, x, u)$ is globally Hölder continuous in t uniformly with respect to $x \in \mathbb{R}$ and u in bounded sets, is globally Lipschitz continuous in x uniformly with respect to $t \in \mathbb{R}$ and u in bounded sets, and is differentiable in u with $f_u(t, x, u)$ being bounded and uniformly continuous in $t \in \mathbb{R}$, $x \in \mathbb{R}$, and u in bounded sets. There are $\beta_0 > 0$ and $P_0 > 0$ such that $f(t, x, u) \leq -\beta_0$ for $t, x \in \mathbb{R}$ and $u \geq P_0$ and $\frac{\partial f}{\partial u}(t, x, u) \leq -\beta_0$ for $t, x \in \mathbb{R}$ and $u \geq 0$. Moreover,

$$-\infty < \inf_{t \in \mathbb{R}, x \in \mathbb{R}, 0 \leq u \leq M} f(t, x, u) \leq \sup_{t \in \mathbb{R}, x \in \mathbb{R}, 0 \leq u \leq M} f(t, x, u) < \infty \quad (1.2)$$

for all $M > 0$, and

$$\liminf_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \inf_{x \in \mathbb{R}} f(\tau, x, 0) d\tau > 0. \quad (1.3)$$

Equations (1.1) appears in the study of population dynamics of species in biology (see [1], [2], [9], [19], [30]), where $u(t, x)$ represents the population density of a species at time t and space location x , $\mathcal{A}u$ describes the dispersal or movement of the organisms and $f(t, x, u)$ describes the growth rate of the population. When $\mathcal{A}u(t, x) = u_{xx}$, it indicates that the movement of the organisms occurs between adjacent locations randomly and the dispersal in this case is referred to as *random dispersal*. When $\mathcal{A}u(t, x) = \int_{\mathbb{R}} \kappa(y - x)u(t, y) dy - u(t, x)$, it indicates that the movement of the organisms occurs between adjacent as well as non-adjacent locations and the dispersal in this case is referred to as *nonlocal dispersal*. The time and space dependence of the equation reflects the heterogeneity of the underlying environments.

Because of biological reason, only nonnegative solutions of (1.1) will be considered throughout this paper. Also, by a solution of (1.1) in this paper, we always mean a classical solution, i.e., a solution satisfies (1.1) in the classical sense, unless otherwise specified. A function $u(t, x)$ is called an *entire solution* if it is a bounded and continuous function on $\mathbb{R} \times \mathbb{R}$ and satisfies (1.1) for $(t, x) \in \mathbb{R} \times \mathbb{R}$. An entire solution $u(t, x)$ is called *strictly positive* if $\inf_{(t, x) \in \mathbb{R} \times \mathbb{R}} u(t, x) > 0$. In the following, we may call a strictly positive entire solution a *positive entire solution* if no confusion occurs.

Equation (1.1) satisfying (H0) is called in literature a Fisher-KPP type equation due to the pioneering papers of Fisher [19] and Kolmogorov, Petrowsky, Piscunov [30] on the following special case of (1.1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u), \quad x \in \mathbb{R}. \quad (1.4)$$

It is clear that $u(t, x) \equiv 1$ is a unique strictly positive entire solution of (1.4). Moreover, it is globally stable with respect to strictly positive initial data $u_0(x)$.

The existence, uniqueness, and stability of positive entire solutions of (1.1) is one of the central problems about the dynamics of (1.1). Assume (H0). We show in this paper that

- (1.1) has a unique stable strictly positive entire solution $u^+(t, x)$. If, in addition, $f(t, x, u)$ is periodic in t and/or x , then so is $u^+(t, x)$, and if $f(t, x, u)$ and $f_u(t, x, u)$ are almost periodic in t and/or x , then so is $u^+(t, x)$ (see Theorem 2.1).

It should be pointed out that when the dispersal is random, there are many studies on the positive entire solutions of (1.1) for various special cases (see [5, 6, 13, 37, 38, 53], etc.). When the dispersal is nonlocal, there are also several studies on the positive entire solutions of (1.1) for some special cases (see [3, 4, 31, 47, 56, 57], etc.) It should also be pointed out that, in the random dispersal case, by the regularity and a priori estimates for parabolic equations (see [21]), it is easy to prove the continuity of bounded solutions. In the nonlocal dispersal case, due to the lack of regularity of solutions, the proof of the continuity of bounded solutions is not trivial. Thanks to the existence, uniqueness, and stability of a unique strictly positive entire solution, (1.1) is also said to be of *monostable type*.

The traveling wave problem is also among the central problems about the dynamics of (1.1). This problem is well understood for the classical Fisher or KPP equation (1.4). For example, Fisher in [19] found traveling wave solutions $u(t, x) = \phi(x - ct)$, ($\phi(-\infty) = 1, \phi(\infty) = 0$) of all speeds $c \geq 2$ and showed that there are no such traveling wave solutions of slower speed. He conjectured that the take-over occurs at the asymptotic speed 2. This conjecture was proved in [30] for some special initial distribution and was proved in [2] for the general case. More precisely, it is proved in [30] that for the nonnegative solution $u(t, x)$ of (1.4) with $u(0, x) = 1$ for $x < 0$ and $u(0, x) = 0$ for $x > 0$, $\lim_{t \rightarrow \infty} u(t, ct)$ is 0 if $c > 2$ and 1 if $c < 2$. It is proved in [2] that for any nonnegative solution $u(t, x)$ of (1.4), if at time $t = 0$, u is 1 near $-\infty$ and 0 near ∞ , then $\lim_{t \rightarrow \infty} u(t, ct)$ is 0 if $c > 2$ and 1 if $c < 2$. Put $c^* = 2$. c^* is of the following spatially spreading property: for any nonnegative solution $u(t, x)$ of (1.4), if at time $t = 0$, $u(0, x) \geq \sigma$ for some $\sigma > 0$ and $x \ll -1$ and $u(0, x) = 0$ for $x \gg 1$, then

$$\inf_{x \leq c't} |u(t, x) - 1| \rightarrow 0 \quad \forall c' < c^* \quad \text{and} \quad \sup_{x \geq c''t} u(t, x) \rightarrow 0 \quad \forall c'' > c^* \quad \text{as} \quad t \rightarrow \infty.$$

In literature, c^* is hence called the *spreading speed* for (1.4). The results on traveling wave solutions of (1.4) have been well extended to general time and space independent monostable equations (see [1], [2], [12], [22], [29], [49], [59], etc.).

Due to the inhomogeneity of the underlying media of biological models in nature, the investigation of the traveling wave problem for time and/or space dependent dispersal evolution equations is gaining more and more attention. The notion of transition waves or generalized traveling waves has been introduced for dispersal evolution equations with general time and space dependence (see Definition 2.2 and Remark 2.2), which naturally extends the notion of traveling wave solutions for time and space independent dispersal evolution equations to the equations with general time and space dependence. A huge amount of research has been carried out toward the transition waves or generalized traveling waves of various time and/or space dependent monostable equations. See, for example, [7, 8, 9, 10, 11, 20, 22, 25, 27, 28, 32, 33, 34, 36, 39, 41, 42, 43, 44, 45, 46, 51, 52, 53, 58, 60, 61, 62, 63], and references therein for space and/or time dependent Fisher-KPP type equations with random dispersal, and see, for example, [15, 16, 17, 35, 48, 54, 55, 56, 57], and references therein for space and/or time dependent Fisher-KPP type equations with nonlocal dispersal.

It should be pointed out that the works [23], [52], [54], and [56] considered the stability of transition waves in spatially periodic and time independent or spatially homogeneous and time dependent Fisher-KPP type equations with random and nonlocal dispersal. In particular, the stability of transition waves in spatially periodic and time independent Fisher-KPP type equations with random dispersal (resp. nonlocal dispersal) is studied in [23] (resp. [56]) and the stability of transition waves in spatially homogeneous and time almost periodic Fisher-KPP type equations with random dispersal (resp. nonlocal dispersal) is investigated in [52] (resp. [54]). The paper [23] also considered the stability of traveling waves in spatially and temporally periodic Fisher-KPP equations with random dispersal (see [23, Section 1.4]).

However, as long as the equations depend on both time and space variables non-periodically, all the existing works are on the existence of transition waves or generalized traveling waves and there is little on the stability of transition waves. In the current paper, we consider the stability of transition waves of Fisher-KPP equations with general time and space dependence (see Definition 2.2 for the definition of transition waves). We show that

- *Any transition wave of (1.1) connecting $u^+(t, x)$ and 0 and satisfying certain decaying property near 0 is asymptotically stable for well-fitted perturbation* (see Theorem 2.2 for detail).

We point out that the existence of transition waves of (1.1) with non-periodic time and/or space dependence has been studied in [35, 41, 42, 43, 52, 53, 54, 63]. Applying the above stability result for general transition waves of (1.1), we prove

- *The non-critical transition waves established in [35, 41, 42, 43, 52, 53, 54, 63] are asymptotically stable for well-fitted perturbation* (see Theorem 2.3 and Remark 2.3 for detail).

Up to the author's knowledge, it is the first time that the stability of transition waves of Fisher-KPP type equations with general time and space dependence is studied. Among the technical tools used in the proofs of the main results are spectral theory for linear dispersal evolution equations with time and space dependence, comparison principle, and the very nontrivial application of the so called part metric.

The rest of the paper is organized as follows. In section 2, we will introduce the standing notations, definitions, and state the main results of the paper. We study the existence, uniqueness and stability of positive entire solutions in section 3. Sections 4 and 5 are devoted to the proofs of the main results on transition waves.

2 Notations, definitions, and main results

In this section, we introduce the standing notations, definitions, and state the main results of the paper. Throughout this section, we assume that (H0) holds.

First of all, we recall the definition of almost periodic functions.

Definition 2.1 (Almost periodic function). (1) A continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ is called almost periodic if for any $\epsilon > 0$, the set

$$T(\epsilon) = \{\tau \in \mathbb{R} \mid |g(t + \tau) - g(t)| < \epsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense in \mathbb{R} .

(2) Let $g(t, x, u)$ be a continuous function of $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$. g is said to be almost periodic in t uniformly with respect to $x \in \mathbb{R}^m$ and u in bounded sets if g is uniformly continuous in $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, and u in bounded sets and for each $x \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$, $g(t, x, u)$ is almost periodic in t .

(3) For a given almost periodic function $g(t, x, u)$, the hull $H(g)$ of g is defined by

$$H(g) = \{\tilde{g}(\cdot, \cdot, \cdot) \mid \exists t_n \rightarrow \infty \text{ such that } g(t + t_n, x, u) \rightarrow \tilde{g}(t, x, u) \text{ uniformly in } t \in \mathbb{R} \text{ and } (x, u) \text{ in bounded sets}\}.$$

Remark 2.1. Let $g(t, x, u)$ be a continuous function of $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$. g is almost periodic in t uniformly with respect to $x \in \mathbb{R}^m$ and u in bounded sets if and only if g is uniformly continuous in $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, and u in bounded sets and for any sequences $\{\alpha'_n\}$, $\{\beta'_n\} \subset \mathbb{R}$, there are subsequences $\{\alpha_n\} \subset \{\alpha'_n\}$, $\{\beta_n\} \subset \{\beta'_n\}$ such that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} g(t + \alpha_n + \beta_m, x, u) = \lim_{n \rightarrow \infty} g(t + \alpha_n + \beta_n, x, u)$$

for each $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ (see [18, Theorems 1.17 and 2.10]).

Next, let

$$C_{\text{unif}}^b(\mathbb{R}) = \{u \in C(\mathbb{R}, \mathbb{R}) \mid u(x) \text{ is uniformly continuous and bounded on } \mathbb{R}\}$$

endowed with the norm $\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x)|$. By general semigroup theory (see [26]), for any $u_0 \in C_{\text{unif}}^b(\mathbb{R})$, (1.1) has a unique (local) solution $u(t, x; t_0, u_0)$ with $u(t_0, x; t_0, u_0) = u_0(x)$ for $x \in \mathbb{R}$.

We then consider the existence, uniqueness, and stability of strictly positive entire solutions of (1.1). The following is the main result of the paper on the existence, uniqueness, and stability of strictly positive entire solution of (1.1).

Theorem 2.1. *There is a unique bounded strictly positive entire solution $u^+(t, x)$ of (1.1) with $u^+(t, x)$ being uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$. Moreover, for any given $u_0 \in C_{\text{unif}}^b(\mathbb{R})$ with $\inf_{x \in \mathbb{R}} u_0(x) > 0$,*

$$\lim_{t \rightarrow \infty} \|u(t + t_0, \cdot; t_0, u_0) - u^+(t + t_0, \cdot)\|_\infty = 0$$

uniformly in $t_0 \in \mathbb{R}$. If, in addition, $f(t, x, u)$ is periodic in t (resp. periodic in x), then so is $u^+(t, x)$. If $f(t, x, u)$ and $f_u(t, x, u)$ are almost periodic in t (resp. almost periodic in x), then so is $u^+(t, x)$.

We now consider transition waves of (1.1) connecting $u^+(t, x)$ and 0.

Definition 2.2. (1) An entire solution $u = U(t, x)$ of (1.1) is called a transition wave (connecting 0 and $u^+(\cdot, \cdot)$) if $U(t, x) \in (0, u^+(t, x))$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, and there exists a function $X : \mathbb{R} \rightarrow \mathbb{R}$, called interface location function, such that

$$\lim_{x \rightarrow -\infty} U(t, x + X(t)) = u^+(t, x + X(t)) \text{ and } \lim_{x \rightarrow \infty} U(t, x + X(t)) = 0 \text{ uniformly in } t \in \mathbb{R}.$$

(2) Assume that $u(t, x) = U(t, x)$ is a transition wave of (1.1) with $X : \mathbb{R} \rightarrow \mathbb{R}$ being an interface location function. $c := \liminf_{t-s \rightarrow \infty, t > s} \frac{X(t) - X(s)}{t - s}$ is called the least mean speed of the transition wave. If $\lim_{t-s \rightarrow \infty, t > s} \frac{X(t) - X(s)}{t - s}$ exists, $c := \lim_{t-s \rightarrow \infty, t > s} \frac{X(t) - X(s)}{t - s}$ is called the average speed or mean speed of the transition wave.

(3) Assume that $u(t, x) = U(t, x)$ is a transition wave of (1.1). It is called asymptotically stable if for any $t_0 \in \mathbb{R}$ and $u_0 \in C_{\text{unif}}^b(\mathbb{R})$ satisfying that $u_0(x) > 0$ for all $x \in \mathbb{R}$ and

$$\inf_{x \leq x_0} u_0(x) > 0 \quad \forall x_0 \in \mathbb{R}, \quad \lim_{x \rightarrow \infty} \frac{u_0(x)}{U(t_0, x)} = 1, \quad (2.1)$$

there holds

$$\lim_{t \rightarrow \infty} \left\| \frac{u(t + t_0, \cdot; t_0, u_0)}{U(t + t_0, \cdot)} - 1 \right\|_{C_{\text{unif}}^b(\mathbb{R})} = 0. \quad (2.2)$$

Remark 2.2. (1) The interface location function $X(t)$ of a transition wave $u = U(t, x)$ tells the position of the transition front of $U(t, x)$ as time t elapses, while the uniform-in- t limits (the essential property in the definition) shows the bounded interface width, that is,

$$\forall 0 < \epsilon_1 \leq \epsilon_2 < 1, \quad \sup_{t \in \mathbb{R}} \text{diam}\{x \in \mathbb{R} | \epsilon_1 \leq U(t, x) \leq \epsilon_2\} < \infty. \quad (2.3)$$

Notice, if $\xi(t)$ is a bounded function, then $X(t) + \xi(t)$ is also an interface location function. Thus, interface location function is not unique. But, it is easy to check that if $Y(t)$ is another interface location function, then $X(t) - Y(t)$ is a bounded function. Hence, interface location functions are unique up to addition by bounded functions and the least mean speed of a transition wave is well defined.

(2) When $f(t + T, x, u) = f(t, x + p, u) = f(t, x, u)$, an entire solution $u = U(t, x)$ of (1.1) is called a periodic traveling wave solution with speed c and connecting $u^+(t, x)$ and 0 if there is $\Phi(x, t, y)$ such that

$$U(t, x) = \Phi(x - ct, t, ct),$$

$$\Phi(x, t + T, y) = \Phi(x, t, y + p) = \Phi(x, t, y),$$

and

$$\lim_{x \rightarrow -\infty} \left(\Phi(x, t, y) - u^+(t, x + y) \right) = 0, \quad \lim_{x \rightarrow \infty} \Phi(x, t, y) = 0$$

uniformly in $t \in \mathbb{R}$ and $y \in \mathbb{R}$. It is clear that if $u = U(t, x)$ is a periodic traveling wave solution, then it is a transition wave.

- (3) When $f(t, x, u)$ is almost periodic in t and periodic in x with period p , an entire solution $u = U(t, x)$ of (1.1) is called an almost periodic traveling wave solution with average speed c and connecting $u^+(t, x)$ and 0 if there are $\xi(t)$ and $\Phi(x, t, y)$ such that

$$U(t, x) = \Phi(x - \xi(t), t, \xi(t)),$$

$\Phi(x, t, y)$ is almost periodic in t and periodic in y ,

$$\lim_{x \rightarrow -\infty} \left(\Phi(x, t, y) - u^+(t, x + y) \right) = 0, \quad \lim_{x \rightarrow \infty} \Phi(x, t, y) = 0$$

uniformly in $t \in \mathbb{R}$ and $y \in \mathbb{R}$, and

$$\lim_{t-s \rightarrow \infty} \frac{\xi(t) - \xi(s)}{t - s} = c.$$

It is clear that if $u = U(t, x)$ is an almost periodic traveling wave solution, then it is a transition wave.

- (4) The reader is referred to [10, 11] for the introduction of the notion of transition waves in the general case, and to [36, 50, 52, 53] for the time almost periodic or space almost periodic cases.
- (5) In the case that $\mathcal{A}u = u_{xx}$, by the regularity and a priori estimates for parabolic equations, any continuous transition wave $u = U(t, x)$ of (1.1) is uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$. In the case that $\mathcal{A}u(t, x) = \int_{\mathbb{R}} \kappa(y - x)u(t, y)dy - u(t, x)$, it is proved in [55] that a transition wave $u = U(t, x)$ of (1.1) is uniformly continuous under quite general conditions.

We have the following general theorem on the stability of transition waves of (1.1).

Theorem 2.2. Assume that $u = U(t, x)$ is a transition wave of (1.1) with interface location $X(t)$ satisfying the following properties: $U(t, x)$ is uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$,

$$\forall \tau > 0, \quad \sup_{t, s \in \mathbb{R}, |t-s| \leq \tau} |X(t) - X(s)| < \infty, \quad (2.4)$$

and there are positive continuous functions $\phi(t, x)$ and $\phi_1(t, x)$ such that

$$\liminf_{x \rightarrow -\infty} \phi(t, x) = \infty, \quad \liminf_{x \rightarrow -\infty} \phi_1(t, x) = \infty, \quad \lim_{x \rightarrow \infty} \phi(t, x) = 0, \quad \lim_{x \rightarrow \infty} \phi_1(t, x) = 0, \quad (2.5)$$

$$\lim_{x \rightarrow -\infty} \frac{\phi(t, x + X(t))}{\phi_1(t, x + X(t))} = 0, \quad \lim_{x \rightarrow \infty} \frac{\phi_1(t, x + X(t))}{\phi(t, x + X(t))} = 0 \quad (2.6)$$

exponentially, and the second limit in (2.6) is uniformly in t ;

$$d^* \phi(t, x) - d_1^* \phi_1(t, x) \leq U(t, x) \leq d^* \phi(t, x) + d_1^* \phi_1(t, x) \quad (2.7)$$

for some $d^*, d_1^* > 0$ and all $t, x \in \mathbb{R}$; and for any given $t_0 \in \mathbb{R}$ and $u_0 \in C_{\text{unif}}^b(\mathbb{R})$ with $u_0(x) \geq 0$, if

$$u_0(x) \geq d\phi(t_0, x) - d_1\phi_1(t_0, x) \quad \left(\text{resp., } u_0(x) \leq d\phi(t_0, x) + d_1\phi_1(t_0, x) \right)$$

for some $0 < d < 2d^*$, $d_1 \gg 1$, and all $x \in \mathbb{R}$, then

$$u(t, x; t_0, u_0) \geq d\phi(t, x) - d_1\phi_1(t, x) \quad \left(\text{resp., } u(t, x; t_0, u_0) \leq d\phi(t, x) + d_1\phi_1(t, x) \right) \quad (2.8)$$

for all $t \geq t_0$ and $x \in \mathbb{R}$. Then the transition wave $u = U(t, x)$ is asymptotically stable.

It should be pointed out that, when $\mathcal{A}u = u_{xx}$, by [24, Proposition 4.2], (2.4) holds for any transition wave of (1.1). In the above theorem, the existence of transition waves is assumed. The existence of transition waves of (1.1) with non-periodic time and/or space dependence has been studied in [35, 41, 42, 43, 52, 53, 54, 63]. Applying Theorem 2.2 or the arguments in the proof of Theorem 2.2, we can establish the asymptotic stability of the transition waves proved in [35, 41, 42, 43, 48, 52, 53, 54, 63]. For convenience, we introduce the following assumptions.

(H1) $f(t, x + p, u) = f(t, x, u)$ for some $p > 0$, $f(t, x, 1) = 0$, $f(t, x, u) \leq f(t, x, 0)$ for $0 \leq u \leq 1$, $\inf_{(t, x) \in \mathbb{R} \times \mathbb{R}} f(t, x, u) > 0$ for $u \in (0, 1)$, and $f(t, x, u) \geq f(t, x, 0)u - Cu^{1+\nu}$ for some $C > 0$, $\delta, \nu \in (0, 1]$, $u \in (0, \delta)$.

(H2) $f(t, x, u) = a(x)(1 - u)$, $\inf_{x \in \mathbb{R}} a(x) > 0$, $a(x)$ is almost periodic in x , and there exists an almost periodic positive function $\phi \in C^2(\mathbb{R})$ such that $\phi_{xx} + a(x)\phi(x) = \lambda_0\phi(x)$ for $x \in \mathbb{R}$, where

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} \mid \exists \phi \in C^2(\mathbb{R}), \phi > 0, \phi_{xx} + a(x)\phi(x) \leq \lambda\phi(x) \text{ for } x \in \mathbb{R}\}.$$

(H3) $f(t, x, u) = f(x, u)$, $f(x, 1) = 0$, $f_u(x, u) < 0$,

$$a(x)g(u) \leq uf(x, u) \leq a(x)u, \quad u \in [0, 1],$$

$$g \in C^1([0, 1]), \quad g(0) = g(1) = 0, \quad g'(0) = 1, \quad 0 \leq g(u) \leq u \text{ for } u \in (0, 1),$$

$$\int_0^1 \frac{u - g(u)}{u^2} du < \infty, \quad \left(\frac{g(u)}{u} \right)' < 0 \text{ for } u \in (0, 1),$$

and

$$0 < a_- := \inf a(x) \leq \sup a(x) := a_+ < \infty.$$

Observe that, assuming one of (H1), (H2), and (H3), $u^+(t, x) \equiv 1$. We have the following theorem on the asymptotic stability of the transition waves established in [41, 42, 43, 48, 52, 53, 63].

Theorem 2.3. (1) Assume that $\mathcal{A}u = u_{xx}$ and that $f(t, x, u)$ satisfies (H1). Then there is $c^* \in \mathbb{R}$ such that for any $c > c^*$, (1.1) has an asymptotically stable transition wave with least mean speed c .

(2) Assume that $\mathcal{A}u = u_{xx}$ and that $f(t, x, u)$ satisfies (H2). Then there is $c^* \in \mathbb{R}$ such that for any $c > c^*$, (1.1) has an asymptotically stable transition wave with mean speed c .

- (3) Assume that $\mathcal{A}u = u_{xx}$ and that $f(t, x, u)$ satisfies (H3). Let $\lambda_0 = \sup \sigma[\partial_{xx}^2 + a(\cdot)]$ and $\lambda_1 = 2a_-$. If $\lambda_0 < \lambda_1$, then for any $\lambda \in (\lambda_0, \lambda_1)$, there is an asymptotically stable transition wave solution $U_\lambda(t, x)$ of (1.1) satisfying that

$$U_\lambda(t, x) \leq v_\lambda(t, x),$$

where $v_\lambda(t, x) = \phi_\lambda(x)e^{\lambda t}$ and $\phi_\lambda(x)$ is the unique solution of

$$\phi_\lambda''(x) + a(x)\phi_\lambda(x) = \lambda\phi_\lambda(x) \quad \text{for } x \in \mathbb{R}, \quad \phi_\lambda(0) = 1, \quad \lim_{x \rightarrow \infty} \phi_\lambda(x) = 0.$$

- (4) Assume that $\mathcal{A}u = \int_{\mathbb{R}} \kappa(y - x)u(t, y)dy - u(t, x)$, and that $f(t, x, u)$ is periodic in both t and x , then there is $c^* > 0$ such that for any $c > c^*$, there is an asymptotically stable periodic traveling wave of (1.1) with speed c .

Remark 2.3. (1) The existence of transition waves in Theorem 2.3(1) is proved in [42]. In fact, the authors of [42] studied a more general case. With the same techniques for the proof of Theorem 2.2 the asymptotic stability of transition waves in this general case can also be established. The following special cases should be mentioned. The existence of transition waves in the case that $f(t, x, u) \equiv f(t, u)$ is proved in [41]. When $f(t, x, u)$ is almost periodic in t or recurrent and unique ergodic in t , the existence of transition waves is proved in [53]. In [52], stability, uniqueness, and almost periodicity of transition waves are also proved when $f(t, x, u) \equiv f(t, u)$ is almost periodic in t .

- (2) The existence of transition waves in Theorem 2.3(2) is proved in [43]. Again, the authors of [43] actually studied a more general case, and with the same techniques for the proof of Theorem 2.2 the asymptotic stability of transition waves in this general case can also be established.
- (3) The existence of transition waves in Theorem 2.3(3) is proved in [63]. By the similar arguments as those in Theorem 2.3(3), it can be proved that the transition waves established in [35] for the case that $\mathcal{A}u = \int_{\mathbb{R}} \kappa(y - x)u(t, y)dy - u(t, x)$ and $f(t, x, u) \equiv f(x, u)$ are asymptotically stable, and that the transition waves established in [54] for the case that $\mathcal{A}u = \int_{\mathbb{R}} \kappa(y - x)u(t, y)dy - u(t, x)$ and $f(t, x, u) \equiv f(t, u)$ are asymptotically stable (the stability of transition waves in this case has been proved in [54] by the “squeezed” techniques).
- (4) The existence of transition waves in Theorem 2.3(4) is proved in [48]. In the case that $f(t, x, u) \equiv f(x, u)$ is periodic in x , the stability of transition waves is proved in [56] by the “squeezed” techniques.
- (5) Transition waves with least mean speed c^* are related to the so called critical traveling waves in literature (see [40], [50]). Both the existence and stability of critical transition waves are much more difficult to study. The reader is referred to [23] and references therein

for the study of the stability of critical traveling waves in the space and/or time periodic or time independent case with random dispersal. The reader is referred to [39, 41, 42, 43] for some results on the existence of critical transition waves of Fisher-KPP equations with random dispersal, and to [17] and reference therein for the existence of critical transition waves in time independent and space periodic or independent Fisher-KPP equations with nonlocal dispersal. The stability of critical transition waves in the general time and space dependent Fisher-KPP equations with random dispersal and in time and/or space periodic Fisher-KPP equations with nonlocal dispersal remains open.

3 Positive entire solutions

In this section, we study the existence, uniqueness, and stability of positive entire solutions and prove Theorem 2.1.

For a given continuous and bounded function $u : [t_1, t_2] \times \mathbb{R} \rightarrow \mathbb{R}$, it is called a *super-solution* (*sub-solution*) of (1.1) on $[t_1, t_2]$ if

$$u_t(t, x) \geq (\leq) (\mathcal{A}u)(t, x) + u(t, x)f(t, x, u(t, x)) \quad \forall (t, x) \in (t_1, t_2) \times \mathbb{R}. \quad (3.1)$$

Proposition 3.1 (Comparison principle). *(1) Suppose that $u^1(t, x)$ and $u^2(t, x)$ are sub- and super-solutions of (1.1) on $[t_1, t_2]$ with $u^1(t_1, x) \leq u^2(t_2, x)$ for $x \in \mathbb{R}$. Then $u^1(t, x) \leq u^2(t, x)$ for $t \in (t_1, t_2)$ and $x \in \mathbb{R}$. Moreover, if $u^1(t_1, x) \not\equiv u^2(t_2, x)$ for $x \in \mathbb{R}$, then $u^1(t, x) < u^2(t, x)$ for $t \in (t_1, t_2)$ and $x \in \mathbb{R}$.*

(2) If $u_{01}, u_{02} \in C_{\text{unif}}^b(\mathbb{R})$ and $u_{01} \leq u_{02}$, then $u(t, \cdot; t_0, u_{01}) \leq u(t, \cdot; t_0, u_{02})$ for $t > t_0$ at which both $u(t, \cdot; t_0, u_{01})$ and $u(t, \cdot; t_0, u_{02})$ exist. Moreover, if $u_{01} \neq u_{02}$, then $u(t, x; t_0, u_{01}) < u(t, x; t_0, u_{02})$ for all $x \in \mathbb{R}$ and $t > t_0$ at which both $u(t, \cdot; t_0, u_{01})$ and $u(t, \cdot; t_0, u_{02})$ exist.

(3) If $u_{01}, u_{02} \in C_{\text{unif}}^b(\mathbb{R})$ and $u_{01} \ll u_{02}$ (i.e. $\inf_{x \in \mathbb{R}} (u_{02}(x) - u_{01}(x)) > 0$), then $u(t, \cdot; t_0, u_{01}) \ll u(t, \cdot; t_0, u_{02})$ for $t > t_0$ at which both $u(t, \cdot; t_0, u_{01})$ and $u(t, \cdot; t_0, u_{02})$ exist.

Proof. When $\mathcal{A}u = u_{xx}$, the proposition follows from comparison principle for parabolic equations (see [21]). When $\mathcal{A}u(t, x) = \int_{\mathbb{R}} \kappa(y - x)u(t, y)dy - u(t, x)$, it follows from comparison principle for nonlocal dispersal evolution equations (see, for example, [57, Propositions 2.1, 2.2]). \square

Note that, by (H0), for $M \gg 1$, $u(t, x) \equiv M$ is a super-solution of (1.1) on \mathbb{R} . The following proposition follows directly from Proposition 3.1 and (H0).

Proposition 3.2. *For any $t_0 \in \mathbb{R}$ and $u_0 \in C_{\text{unif}}^b(\mathbb{R})$ with $u_0(\cdot) \geq 0$, $u(t, x; t_0, u_0)$ exists for all $t \geq t_0$ and $\sup_{t \geq 0} \|u(t + t_0, \cdot; t_0, u_0)\|_{\infty} < \infty$.*

For given $u, v \in C_{\text{unif}}^b(\mathbb{R})$ with $u, v \geq 0$, if there is $\alpha_0 \geq 1$ such that

$$\frac{1}{\alpha_0}v(x) \leq u(x) \leq \alpha_0 v(x) \quad \forall x \in \mathbb{R},$$

then we can define the so called *part metric* $\rho(u, v)$ between u and v by

$$\rho(u, v) = \inf\{\ln \alpha \mid \alpha \geq 1, \frac{1}{\alpha}v(\cdot) \leq u(\cdot) \leq \alpha v(\cdot)\}.$$

Note that for any given $u, v \in C_{\text{unif}}^b(\mathbb{R})$ with $u, v \geq 0$, the part metric between u and v may not be defined.

Proposition 3.3. (1) For given $u_0, v_0 \in C_{\text{unif}}^b(\mathbb{R})$ with $u_0, v_0 \geq 0$, if $\rho(u_0, v_0)$ is defined, then for any $t_0 \in \mathbb{R}$, $\rho(u(t+t_0, \cdot; t_0, u_0), u(t+t_0, \cdot; t_0, v_0))$ is also defined for all $t > 0$. Moreover, $\rho(u(t+t_0, \cdot; t_0, u_0), u(t+t_0, \cdot; t_0, v_0))$ is non-increasing in t .

(2) For any $\epsilon > 0$, $\sigma > 0$, $M > 0$, and $\tau > 0$ with $\epsilon < M$ and $\sigma \leq \ln \frac{M}{\epsilon}$, there is $\delta > 0$ such that for any $u_0, v_0 \in C_{\text{unif}}^b(\mathbb{R})$ with $\epsilon \leq u_0(x) \leq M$, $\epsilon \leq v_0(x) \leq M$ for $x \in \mathbb{R}$ and $\rho(u_0, v_0) \geq \sigma$, there holds

$$\rho(u(\tau+t_0, \cdot; t_0, u_0), u(\tau+t_0, \cdot; t_0, v_0)) \leq \rho(u_0, v_0) - \delta \quad \text{for all } t_0 \in \mathbb{R}.$$

Proof. It follows from the similar arguments as those in [31, Proposition 3.4]. \square

Proof of Theorem 2.1. We divide the proof into two steps.

Step 1. In this step, we prove the existence, uniqueness, and stability of bounded strictly positive entire solutions $u^+(t, x)$ with $u^+(t, x)$ being uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Note that the existence, uniqueness, and stability of bounded strictly positive entire solutions follows from the similar arguments as those in [14, Theorem 1.1(1)]. We outline the proof of existence in the following for the use in the proof of uniform continuity.

Clearly, $u(t, x) \equiv M$ is a super-solution of (1.1) for any $M \gg 1$. For given $\delta > 0$, let $v_\delta \equiv \delta$. By the similar arguments of [14, Theorem 1.1(1)], there are $\delta_0 > 0$ and $T > 0$ such that for any $0 < \delta \leq \delta_0$,

$$u(t_0 + T, \cdot; t_0, v_\delta) \geq v_\delta \quad \forall t_0 \in \mathbb{R}. \quad (3.2)$$

Fix $M \gg 1$ and $0 < \delta \ll 1$. Let $u_M \equiv M$ and $u_\delta \equiv \delta$. For given $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, let $u_{m-n, m}(\cdot) = u(mT, \cdot; (m-n)T, u_\delta)$ and $u^{m-n, m}(\cdot) = u(mT, \cdot; (m-n)T, u_M)$. Then for any $m \in \mathbb{Z}$ and $n \geq 0$,

$$\delta \leq u_{m-1, m}(\cdot) \leq u_{m-2, m}(\cdot) \leq \dots, \quad M \geq u^{m-1, m}(\cdot) \geq u^{m-2, m}(\cdot) \geq \dots.$$

Moreover, by Proposition 3.3, it is not difficult to prove that

$$\rho(u^{m-n, m}, u_{m-n, m}) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{uniformly in } m \in \mathbb{Z}. \quad (3.3)$$

Hence there are $u^{*, m} \in C_{\text{unif}}^b(\mathbb{R})$ ($m \in \mathbb{Z}$) such that

$$u^{*, m}(x) = \lim_{n \rightarrow \infty} u_{m-n, m}(x) = \lim_{n \rightarrow \infty} u^{m-n, m}(x) \quad \text{uniformly in } x \in \mathbb{R}.$$

It is then not difficult to see that $u(t, \cdot; 0, u^{*,0})$ has backward extension for all $t < 0$ and hence $u^+(t, x) := u(t, \cdot; 0, u^{*,0})$ is an entire solution. Moreover,

$$\delta \leq u^+(kT, x) \leq M \quad \forall x \in \mathbb{R}, k \in \mathbb{Z} \quad (3.4)$$

and

$$u^+(kT, x) = \lim_{n \rightarrow \infty} u(kT, x; (k-n)T, u_M) \quad \text{uniformly in } x \in \mathbb{R}, k \in \mathbb{Z}. \quad (3.5)$$

By (3.4), $u^+(t, x)$ is a bounded strictly positive entire solution of (1.1).

We now prove the uniform continuity of $u^+(t, x)$ in $(t, x) \in \mathbb{R} \times \mathbb{R}$. In the case that $\mathcal{A}u = u_{xx}$, by the regularity and a priori estimates for parabolic equations, we have that $u^+(t, x)$ is uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$. We then only need to prove that $u^+(t, x)$ is uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$ in the case that \mathcal{A} is a nonlocal dispersal operator. In this case, by the boundedness of $u^+(t, x)$, $u_t^+(t, x)$ is uniformly bounded. This implies that $u^+(t, x)$ is uniformly continuous in t uniformly with respect to $x \in \mathbb{R}$. We claim that $u^+(t, x)$ is also uniformly continuous in x uniformly with respect to $t \in \mathbb{R}$. Indeed, By (3.5), for any $\epsilon > 0$, there is $K \in \mathbb{N}$ such that

$$|u^+(t + kT, x) - u(t + kT, x; (k - K)T, u_M)| < \epsilon \quad \forall x \in \mathbb{R}, 0 \leq t \leq T, k \in \mathbb{Z}. \quad (3.6)$$

It then suffices to prove that $u(t + kT, x; (k - K)T, u_M)$ is uniformly continuous in x uniformly with respect to $t \in [0, T]$ and $k \in \mathbb{Z}$. For any given $h > 0$ and $k \in \mathbb{Z}$, let

$$v(t, x; h, k) = u(t, x + h; (k - K)T, u_M) - u(t, x; (k - K)T, u_M).$$

Then $v(t, x; h, k)$ satisfies

$$\begin{cases} v_t = \mathcal{A}v(t, x) + p(t, x)v(t, x; h, k) + q(t, x), & x \in \mathbb{R}, t > (k - K)T \\ v((k - K)T, x; h, k) = 0, & x \in \mathbb{R}, \end{cases}$$

where

$$\begin{aligned} p(t, x) &= f(t, x + h, u(t, x + h; (k - K)T, u_M)) - u(t, x; (k - K)T, u_M) \\ &\quad - \int_0^1 f_u(t, x, s u(t, x + h; (k - K)T, u_M) + (1 - s)u(t, x; (k - K)T, u_M)) ds \end{aligned}$$

and

$$q(t, x) = u(t, x; (k - K)T, u_M) [f(t, x + h, u(t, x + h; (k - K)T, u_M)) - f(t, x, u(t, x + h; (k - K)T, u_M))].$$

Then

$$v(t, \cdot; h, k) = \int_{(k-K)T}^t e^{\mathcal{A}(t-\tau)} p(\tau, \cdot) v(\tau, \cdot; h, k) d\tau + \int_{(k-K)T}^t e^{\mathcal{A}(t-\tau)} q(\tau, \cdot) d\tau.$$

This together with the assumption (H0) implies that there is $C > 0$ such that

$$|v(t, x; h, k)| \leq Ch \quad \forall 0 \leq t \leq T, x \in \mathbb{R}, k \in \mathbb{Z}.$$

Then by (3.6) and Gronwall's inequality, $u^+(t, x)$ is uniformly continuous in x uniformly with respect to $t \in \mathbb{R}$ and then $u^+(t, x)$ is uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$.

By the similar arguments as those in [14, Theorem 1.1(1)], we have that bounded strictly positive entire solutions of (1.1) are stable and unique. This completes the proof of Step 1.

Step 2. In this step, we show that $u^+(t, x)$ is almost periodic in t (resp., in x) if $f(t, x, u)$ is almost periodic in t (resp., in x).

Assume that $f(t, x, u)$ is almost periodic in t . For any given sequences $\{\alpha'_n\}, \{\beta'_n\} \subset \mathbb{R}$, there are subsequences $\{\alpha_n\} \subset \{\alpha'_n\}, \{\beta_n\} \subset \{\beta'_n\}$ such that

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} f(t + \alpha_n + \beta_m, x, u) = \lim_{n \rightarrow \infty} f(t + \alpha_n + \beta_n, x, u).$$

By the uniform continuity of $u^+(t, x)$, without loss of generality, we may assume that $\lim_{n \rightarrow \infty} u^+(t + \alpha_n, x)$ exists locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}$. Let

$$\hat{f}(t, x, u) = \lim_{n \rightarrow \infty} f(t + \alpha_n, x, u) \quad \text{and} \quad \hat{u}^+(t, x) = \lim_{n \rightarrow \infty} u^+(t + \alpha_n, x).$$

It is clear that $\hat{f}(t, x, u)$ satisfies (H0) and $\hat{u}^+(t, x)$ is uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$ and $\inf_{t \in \mathbb{R}, x \in \mathbb{R}} \hat{u}^+(t, x) > 0$. Moreover, $\hat{u}^+(t, x)$ is a bounded positive entire solution of (1.1) with f being replaced by \hat{f} . Similarly, without loss of generality, we may assume that $\lim_{m \rightarrow \infty} \hat{u}^+(t + \beta_m, x)$ and $\lim_{n \rightarrow \infty} u^+(t + \alpha_n + \beta_n, x)$ exist locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}$. Let

$$\check{f}(t, x, u) = \lim_{m \rightarrow \infty} \hat{f}(t + \beta_m, x, u), \quad \check{f}(t, x, u) = \lim_{n \rightarrow \infty} f(t + \alpha_n + \beta_n, x, u)$$

and

$$\check{u}^+(t, x) = \lim_{m \rightarrow \infty} \hat{u}^+(t + \beta_m, x, u), \quad \tilde{u}^+(t, x) = \lim_{n \rightarrow \infty} u^+(t + \alpha_n + \beta_n, x).$$

Then $\check{f}(t, x, u)$ and $\tilde{f}(t, x, u)$ satisfy (H0), $\check{u}^+(t, x)$ and $\tilde{u}^+(t, x)$ are uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\check{u}^+(t, x)$ is a bounded positive entire solution of (1.1) with f being replaced by \check{f} , and $\tilde{u}^+(t, x)$ is a bounded positive entire solution of (1.1) with f being replaced by \tilde{f} . Note that $\tilde{f} = \check{f}$. Then by the uniqueness of positive entire solutions of (1.1) with f being replaced by \check{f} , we have

$$\check{u}^+(t, x) = \tilde{u}^+(t, x).$$

This together with Remark 2.1 implies that $u^+(t, x)$ is almost periodic in t uniformly with respect to $x \in \mathbb{R}$. In particular, if f is periodic in t , so is $u^+(t, x)$.

Assume that $f(t, x, u)$ is almost periodic in x . Similarly, we can prove that $u^+(t, x)$ is almost periodic in x uniformly with respect to $t \in \mathbb{R}$. In particular, if f is periodic in x , so is $u^+(t, x)$. The theorem is thus proved. \square

4 Stability of transition waves

In this section, we study the stability of transition waves in the general case and prove Theorem 2.2.

Proof of Theorem 2.2. Suppose that $u = U(t, x)$ is a transition wave of (1.1) satisfying (2.4) and (2.7).

Note that, for given $u_0(\cdot)$ satisfying (2.1) and given $t_0 \in \mathbb{R}$, the part metric $\rho(u_0, U(t_0))$ is well defined and then $\rho(u(t, \cdot; t_0, u_0), U(t + t_0, \cdot))$ is well defined for all $t \geq 0$. To prove (2.2), it suffices to prove that for any $\epsilon > 0$, there is $T > 0$ such that

$$\rho(u(t + t_0, \cdot; t_0, u_0), U(t + t_0, \cdot)) < \epsilon \quad \text{for all } t \geq T. \quad (4.1)$$

In fact, by Proposition 3.3(1), we only need to prove that for any $\epsilon > 0$,

$$\rho(u(t + t_0, \cdot; t_0, u_0), U(t + t_0, \cdot)) < \epsilon \quad \text{for some } t > 0. \quad (4.2)$$

Assume by contradiction that there is $\epsilon_0 > 0$ such that

$$\rho(u(t + t_0, \cdot; t_0, u_0), U(t + t_0, \cdot)) \geq \epsilon_0 \quad (4.3)$$

for all $t \geq 0$. Fix a $\tau > 0$. We claim that if (4.3) holds, then there is $\delta > 0$ such that

$$\rho(u(\tau + s + t_0, \cdot; t_0, u_0), U(\tau + s + t_0, \cdot)) \leq \rho(u(s + t_0, \cdot; t_0, u_0), U(s + t_0, \cdot)) - \delta \quad (4.4)$$

for all $s \geq 0$.

Before proving (4.4), we prove that (4.4) gives rise to a contradiction. In fact, assume (4.4). Then we have

$$\rho(u(n\tau + t_0, \cdot; t_0, u_0), U(n\tau + t_0, \cdot)) \leq \rho(u(t_0, \cdot; t_0, u_0), U(t_0, \cdot)) - n\delta$$

for all $n \geq 0$. Letting $n \rightarrow \infty$, we have $\rho(u(n\tau + t_0, \cdot; t_0, u_0), U(n\tau + t_0, \cdot)) \rightarrow -\infty$, which is a contradiction. Therefore, (4.3) does not hold and then for any $\epsilon > 0$,

$$\rho(u(t + t_0, \cdot; t_0, u_0), U(t + t_0, \cdot)) < \epsilon \quad \text{for some } t > 0.$$

This together with Proposition 3.3 implies that

$$\lim_{t \rightarrow \infty} \rho(u(t + t_0, \cdot; t_0, u_0), U(t + t_0, \cdot)) = 0.$$

The theorem then follows.

We now prove that if (4.3) holds, then there is $\delta > 0$ such that (4.4) holds.

First, for any $0 < \epsilon < \frac{\epsilon_0}{4+2\epsilon_0}$, by (2.1), (2.5), and (2.6), there is $d_1 \gg d_1^*$ such that

$$d^*(1 - \epsilon)\phi(t_0, x) - d_1\phi_1(t_0, x) \leq u_0(x) \leq d^*(1 + \epsilon)\phi(t_0, x) + d_1\phi_1(t_0, x).$$

By (2.8), there holds

$$d^*(1 - \epsilon)\phi(t, x) - d_1\phi_1(t, x) \leq u(t, x; t_0, u_0) \leq d^*(1 + \epsilon)\phi(t, x) + d_1\phi_1(t_0, x) \quad \forall t \geq t_0.$$

Then by (2.6) and (2.7), for $x - X(t) \gg 1$, we have that $0 < \frac{\phi_1(t, x)}{\phi(t, x)} \ll 1$,

$$\begin{aligned} u(t, x; t_0, u_0) &\leq d^*(1 + \epsilon)\phi(t, x) \left(1 + \frac{d_1}{d^*(1 + \epsilon)} \frac{\phi_1(t, x)}{\phi(t, x)}\right) \\ &\leq (1 + \epsilon)U(t, x) \left(1 + \frac{d_1}{d^*(1 + \epsilon)} \frac{\phi_1(t, x)}{\phi(t, x)}\right) \left(1 - \frac{d_1^*}{d^*} \frac{\phi_1(t, x)}{\phi(t, x)}\right)^{-1}, \end{aligned}$$

and

$$\begin{aligned} u(t, x; t_0, u_0) &\geq d^*(1 - \epsilon)\phi(t, x) \left(1 - \frac{d_1}{d^*(1 - \epsilon)} \frac{\phi_1(t, x)}{\phi(t, x)}\right) \\ &\geq (1 - \epsilon)U(t, x) \left(1 - \frac{d_1}{d^*(1 - \epsilon)} \frac{\phi_1(t, x)}{\phi(t, x)}\right) \left(1 + \frac{d_1^*}{d^*} \frac{\phi_1(t, x)}{\phi(t, x)}\right)^{-1}. \end{aligned}$$

This together with (2.4) and (2.6) implies that, for any $s \geq 0$, there is $x_s (\geq X(t_0 + s))$ such that

$$\sup_{s \geq 0, t \in [t_0 + s, t_0 + s + \tau]} |x_s - X(t)| < \infty \quad (4.5)$$

and

$$\frac{1}{1 + \epsilon_0/2} U(t, x) \leq u(t, x; t_0, u_0) \leq (1 + \epsilon_0/2)U(t, x) \quad \forall t \in [t_0 + s, t_0 + s + \tau], x \geq x_s. \quad (4.6)$$

Next, we claim that

$$\inf_{s \geq 0, t \in [s + t_0, \tau + s + t_0], x \leq x_s} U(t, x) > 0. \quad (4.7)$$

In fact, if this is not true, then there are $s_n \geq 0$, $t_n \in [s_n + t_0, \tau + s_n + t_0]$, and $x_n \leq x_{s_n}$ such that

$$\lim_{n \rightarrow \infty} U(t_n, x_n) = 0.$$

By Remark 2.2(1) and (4.5), there are $\beta > 0$ and $\tilde{x}_n \leq x_n$ such that

$$U(s_n + t_0, \tilde{x}_n) \geq \beta \quad \text{and} \quad \sup_{n \geq 1} |x_{s_n} - \tilde{x}_n| < \infty.$$

Let

$$U_n(t, x) = U(t + s_n + t_0, x + \tilde{x}_n).$$

By the uniform continuity of $U(t, x)$ in $(t, x) \in \mathbb{R} \times \mathbb{R}$, without loss of generality, we may assume that

$$\lim_{n \rightarrow \infty} U_n(t, x) = U^*(t, x)$$

locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}$. Without loss of generality, we may also assume that

$$\lim_{n \rightarrow \infty} (t_n - s_n - t_0) = t^*, \quad \lim_{n \rightarrow \infty} (x_n - \tilde{x}_n) = x^*,$$

and

$$\lim_{n \rightarrow \infty} f(t + s_n + t_0, x + \tilde{x}_n, u) = f^*(t, x, u)$$

locally uniformly in $(t, x, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Then

$$U^*(t, x) \geq 0, \quad U^*(t^*, 0) \geq \beta, \quad U^*(t^*, x^*) = 0, \quad (4.8)$$

and $U^*(t, x)$ is a solution of (1.1) with $f(t, x, u)$ being replaced by $f^*(t, x, u)$. Then by comparison principle, we have either $U^*(t, x) \equiv 0$ or $U^*(t, x) > 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, which contradicts to (4.8). Therefore, (4.7) holds.

Now for given $s \geq 0$, let

$$\rho(s + t_0) = \rho(u(s + t_0, \cdot; t_0, u_0), U(s + t_0, \cdot)).$$

By (4.3),

$$\rho(t_0) \geq \rho(s + t_0) \geq \epsilon_0 \quad (4.9)$$

and

$$\frac{1}{\rho(s + t_0)} U(s + t_0, \cdot) \leq u(s + t_0, \cdot; t_0, u_0) \leq \rho(s + t_0) U(s + t_0, \cdot). \quad (4.10)$$

It follows from (4.10) and Proposition 3.1 that

$$u(t + s + t_0, \cdot; t_0, u_0) \leq u(t + s + t_0, \cdot; s + t_0, \rho(s + t_0) U(s + t_0, \cdot)) \quad \text{for } t \geq 0.$$

Let

$$\hat{u}(t, x) = u(t + s + t_0, \cdot; s + t_0, \rho(s + t_0) U(s + t_0, \cdot)),$$

$$\tilde{u}(t, x) = \rho(s + t_0) u(t + s + t_0, \cdot; s + t_0, U(s + t_0, \cdot)) (= \rho(s + t_0) U(t + s + t_0, x)),$$

and

$$\bar{u}(t, x) = \tilde{u}(t, x) - \hat{u}(t, x).$$

Then

$$\begin{aligned} \bar{u}_t(t, x) &= \mathcal{A}\bar{u}(t, x) + \tilde{u}(t, x) f(t + s + t_0, x, U(t + s + t_0, x)) - \hat{u}(t, x) f(t + s + t_0, x, \hat{u}(t, x)) \\ &= \mathcal{A}\bar{u}(t, x) + p(t, x) \bar{u}(t, x) + b(t, x), \end{aligned} \quad (4.11)$$

where

$$p(t, x) = f(t + s + t_0, x, \hat{u}(t, x)) + \tilde{u}(t, x) \int_0^1 f_u(t + s + t_0, x, r\tilde{u}(t, x) + (1 - r)\hat{u}(t, x)) dr,$$

and

$$\begin{aligned} b(t, x) &= \tilde{u}(t, x) [f(t + s + t_0, x, U(t + s + t_0, x)) - f(t + s + t_0, x, \tilde{u}(t, x))] \\ &= \tilde{u}(t, x) [f(t + s + t_0, x, U(t + s + t_0, x)) - f(t + s + t_0, x, \rho(s + t_0) U(t + s + t_0, x))]. \end{aligned}$$

By (H0) and (4.7), there is $b_0 > 0$ such that for any $s \geq 0$,

$$\inf_{t \in [s + t_0, \tau + s + t_0], x \leq x_s} b(t, x) \geq b_0 > 0. \quad (4.12)$$

In the case that $\mathcal{A}u = u_{xx}$, by (4.11) and comparison principle for parabolic equations, we have

$$\bar{u}(\tau, \cdot) \geq \int_{t_0+s}^{t_0+s+\tau} e^{(\tau+s+t_0-r)p_{\inf}} T(\tau+s+t_0-r)b(r, \cdot)dr,$$

where $p_{\inf} = \inf_{t \in [t_0+s, t_0+s+\tau], x \in \mathbb{R}} p(t, x)$,

$$T(t)u(x) = \int_{\mathbb{R}^N} G(x-y, t)u(y)dy$$

and

$$G(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}}. \quad (4.13)$$

This together with (4.12) implies that there is $\delta_0 > 0$ such that for any $s \geq 0$,

$$\bar{u}(\tau, x) \geq \delta_0 \quad \forall x \leq x_s$$

and then

$$u(\tau+s+t_0, x; t_0, u_0) \leq \rho(s+t_0)U(t+s+t_0, x) - \delta_0 \quad \forall x \leq x_s. \quad (4.14)$$

By (4.6) and (4.14), then there is $\delta_1 > 0$ such that

$$u(\tau+s+t_0, \cdot; t_0, u_0) \leq (\rho(s+t_0) - \delta_1)U(\tau+s+t_0, \cdot) \quad \text{for all } s \geq 0.$$

Similarly, we can prove that there is $\delta_2 > 0$ such that

$$\frac{1}{\rho(s+t_0) - \delta_2} U(\tau+s+t_0, \cdot) \leq u(\tau+s+t_0, \cdot; t_0, u_0) \quad \text{for all } s \geq 0.$$

The claim (4.4) then holds for $\delta = \min\{\delta_1, \delta_2\}$.

In the case that $\mathcal{A}u(t, x) = \int_{\mathbb{R}} \kappa(y-x)u(t, y)dy - u(t, x)$, we have that $\bar{u}(t, x)$ satisfies

$$\bar{u}_t = \int_{\mathbb{R}} \kappa(y-x)\bar{u}(t, y)dy - \bar{u}(t, x) + p(t, x)\bar{u}(t, x) + b(t, x) \quad \forall x \in \mathbb{R}.$$

Note that

$$\int_{\mathbb{R}} \kappa(y-x)\bar{u}(t+s+t_0, y)dy \geq 0.$$

Hence for $x \leq x_s$,

$$\bar{u}_t(t, x) \geq (-1 + p(t, x))\bar{u}(t, x) + b_0.$$

This implies that

$$\bar{u}(\tau, x) \geq \int_{s+t_0}^{\tau+s+t_0} e^{(-1+p_{\inf})(\tau+s+t_0-r)} b_0 dr \quad \forall x \leq x_s.$$

Then by the similar arguments as in the above, the claim (4.4) then holds for some $\delta > 0$. \square

5 Existence and stability of transition waves

In this section, we consider the stability of transition waves of (1.1) established in literature and prove Theorem 2.3.

Proof of Theorem 2.3 (1). As it is mentioned in Remark 2.3(1), the existence of transition waves is established in [42]. In the following, we outline the construction of transition waves from [42] and show that they satisfy the conditions in Theorem 2.2 and hence are asymptotically stable.

Assume (H1) and let $a(t, x) = f(t, x, 0)$. For any $\mu > 0$, by [42, Lemma 3.1], the equation

$$u_t = u_{xx} + a(t, x)u$$

has a positive solution of the form

$$u_\mu(t, x) = e^{-\mu x} \eta_\mu(t, x), \quad \text{where} \quad \eta_\lambda(t, x + p) = \eta_\lambda(t, x).$$

By Lemma [42, Lemma 3.2], there are $\beta > 0$ and a uniformly Lipschitz continuous function $S_\mu(t)$ such that

$$|S_\mu(t) - \frac{1}{\mu} \ln \|\eta_\mu(\cdot, t)\|_{L^\infty}| \leq \beta \quad \forall t \in \mathbb{R}.$$

Let

$$c_\mu = \liminf_{t-s \rightarrow \infty} \frac{S_\mu(t) - S_\mu(s)}{t - s}.$$

By [42, Lemma 3.4], there is $\mu^* > 0$ such that for $0 < \mu < \mu^*$, c_μ is decreasing for $\mu \in (0, \mu^*)$ and this does not hold for $\mu \in (0, \tilde{\mu}^*)$ for any $\tilde{\mu}^* > \mu^*$. Let $c^* = c_{\mu^*}$. Note that when $a(x) \equiv a$, $c^* = 2\sqrt{a}$. We show that Theorem 2.3(1) holds with this c^* .

To this end, for fixed $\mu > 0$, let

$$\tilde{\phi}_\mu(t, x) = e^{-\mu S_\mu(t)} \eta_\mu(t, x).$$

By equation (38) in [42], there is $C_\mu > 0$ such that

$$C_\mu \leq \tilde{\phi}_\mu(t, x) \leq e^{\mu\beta} \quad \forall x \in \mathbb{R}, t \in \mathbb{R}.$$

Note that for any $c > c^*$, there is $0 < \mu < \mu^*$ such that $c_\mu = c$. Let μ' be such that $\mu < \mu' < (1 + \nu)\mu$. By [41, Lemma 3.2], there is $\sigma(\cdot) \in W^{1,\infty}(\mathbb{R})$ such that

$$\inf\{\sigma'(t) + \mu'(c_\mu(t) - c_{\mu'}(t))\} > 0,$$

where $c_\mu(t) = S'_\mu(t)$ and $c_{\mu'}(t) = S'_{\mu'}(t)$. Let

$$\phi(t, x) = e^{-\mu(x - S_\mu(t))} \tilde{\phi}_\mu(t, x) \quad \text{and} \quad \phi_1(t, x) = e^{\sigma(t) - \mu'(x - S_\mu(t))} \tilde{\phi}_{\mu'}(t, x).$$

By the arguments of [42, Theorem 2.1], for any $u_0 \in C_{\text{unif}}^b(\mathbb{R})$ ($u_0(x) \geq 0$) and $t_0 \in \mathbb{R}$ with

$$u_0(x) \leq d\phi(t_0, x) + d_1\phi_1(t_0, x) \quad (\text{resp. } u_0(x) \geq d\phi(t_0, x) - d_1\phi_1(t_0, x)) \quad \forall x \in \mathbb{R}$$

for some $d > 0$ and $d_1 > 0$ (resp., for some $d > 0$ and $d_1 \gg 1$), there holds

$$u(t, x; t_0, u_0) \leq d\phi(t, x) + d_1\phi_1(t, x) \quad (\text{resp. } u(t, x; t_0, u_0) \geq d\phi(t, x) - d_1\phi_1(t, x)) \quad \forall x \in \mathbb{R}$$

for all $t \geq t_0$. Moreover, there is a transition wave solution $u = U_\mu(t, x)$ of (1.1) satisfying that

$$\phi(t, x) - d_\mu\phi_1(t, x) \leq U_\mu(t, x) \leq \phi(t, x) \quad \forall t, x \in \mathbb{R}$$

for some $d_\mu \gg 1$. Clearly, $X(t) = S_\mu(t)$ is an interface location of $u = U_\mu(t, x)$ satisfying (2.4), and $\phi(t, x)$ and $\phi_1(t, x)$ satisfy (2.5) and (2.6). It then follows from Theorem 2.2 that $u = U_\lambda(t, x)$ is asymptotically stable. Clearly, $u = U_\mu(t, x)$ has least mean speed c . Theorem 2.3(1) thus follows. \square

Proof of Theorem 2.3 (2). As it is mentioned in Remark 2.3(2), the existence of transition waves is established in [43]. Similarly, we outline the construction of transition waves from [43] and show that they satisfy the conditions in Theorem 2.2 and hence are asymptotically stable.

Assume (H2) and let $a(x) = f(x, 0)$. By [43, Proposition 1.3], for any $\lambda > \lambda_0$, there is a unique positive $\phi_\lambda \in C^2(\mathbb{R})$ such that

$$\phi_\lambda'' + a(x)\phi_\lambda(x) = \lambda\phi_\lambda(x) \quad \text{in } \mathbb{R}, \quad \phi_\lambda(0) = 1, \quad \lim_{x \rightarrow \infty} \phi_\lambda(x) = 0,$$

and there exists the limit

$$\mu(\lambda) := - \lim_{x \rightarrow \pm\infty} \frac{1}{x} \ln \phi_\lambda(x) > 0. \quad (5.1)$$

By [43, Lemma 2.3], $\phi_\lambda(x)$ is unbounded, and by [43, Lemma 2.4], $\phi_\lambda'(x)/\phi_\lambda(x)$ is almost periodic in x . Let

$$c^* = \inf_{\lambda > \lambda_0} \frac{\lambda}{\mu(\lambda)}.$$

In the following we show that Theorem 2.3(2) holds with this c^* .

By [43, Lemma 3.2], for any $c > c^*$, there is $\lambda > \lambda_0$ such that

$$c = \frac{\lambda}{\mu(\lambda)} \quad \text{and} \quad c > \frac{\lambda'}{\mu(\lambda')} \quad \text{for } \lambda' - \lambda > 0 \text{ small enough.}$$

Let $\sigma_\lambda(x) = -\frac{\phi_\lambda'(x)}{\phi_\lambda(x)}$. By [43, Proposition 3.3], there exist $\delta > 0$, $\epsilon \in (0, 1)$, and a function $\theta \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\inf_{x \in \mathbb{R}} \theta(x) > 0, \quad -\theta'' + 2\sigma_\lambda\phi' - (\sigma_\lambda^2 - \sigma_\lambda' + a)\theta \geq (\delta - (1 + \epsilon)\lambda)\theta \quad \text{in } \mathbb{R}.$$

By the arguments of [43, Proposition 3.4], for any $t_0 \in \mathbb{R}$ and $u_0 \in C_{\text{unif}}^b(\mathbb{R})$ ($u_0 \geq 0$) with

$$u_0(x) \leq d\phi_\lambda(x)e^{\lambda t_0} + d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t_0}$$

$$(\text{resp.}, u_0(x) \geq d\phi_\lambda(x)e^{\lambda t_0} - d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t_0})$$

for some $d > 0$ and $d_1 > 0$ (resp., for $d > 0$ and $d_1 \gg 1$), there holds

$$\begin{aligned} u(t, x; t_0, u_0) &\leq d\phi_\lambda(x)e^{\lambda t} + d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t} \\ (\text{resp., } u(t, x; t_0, u_0) &\geq d\phi_\lambda(x)e^{\lambda t} - d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t}) \end{aligned}$$

for all $t \geq t_0$. Moreover, there is a transition wave $u(t, x) = U_\lambda(t, x)$ of (1.1) with mean speed c and satisfying that

$$\phi_\lambda(x)e^{\lambda t} - d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t} \leq U_\lambda(t, x) \leq \phi_\lambda(x)e^{\lambda t}$$

for some $d_1 \gg 1$.

Let $X(t)$ be such that $\phi_\lambda(X(t)) = e^{-\lambda t}$, that is, $\int_0^{X(t)} \frac{\phi'_\lambda(y)}{\phi_\lambda(y)} dy = -\lambda t$. By [43, Lemma 3.5], $X(t)$ is an interface location of $u = U_\lambda(t, x)$. By [24, Proposition 4.2], $X(t)$ satisfies (2.4). Let

$$\phi(t, x) = \phi_\lambda(x)e^{\lambda t}, \quad \phi_1(t, x) = \theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t}.$$

It is clear that $\phi(t, x)$ and $\phi_1(t, x)$ satisfy (2.5). Note that

$$\frac{\phi_1(t, x + X(t))}{\phi(t, x + X(t))} = \frac{\left(e^{\int_0^{x+X(t)} \frac{\phi'_\lambda(y)}{\phi_\lambda(y)} dy} e^{\lambda t} \right)^{1+\epsilon}}{e^{\int_0^{x+X(t)} \frac{\phi'_\lambda(y)}{\phi_\lambda(y)} dy} e^{\lambda t}} = e^{\epsilon \int_{X(t)}^{x+X(t)} \frac{\phi'_\lambda(y)}{\phi_\lambda(y)} dy}. \quad (5.2)$$

By the almost periodicity of $\frac{\phi'_\lambda(x)}{\phi_\lambda(x)}$, (5.1), and (5.2),

$$\lim_{|x| \rightarrow \infty} \frac{1}{x} \int_{X(t)}^{x+X(t)} \frac{\phi'_\lambda(y)}{\phi_\lambda(y)} dy = -\mu(\lambda) < 0$$

uniformly in $t \in \mathbb{R}$. This implies that $\phi_1(t, x)$ and $\phi_2(t, x)$ satisfy (2.6). It then follows from Theorem 2.2 that $u = U_\lambda(t, x)$ is an asymptotically stable transition wave of (1.1) with average speed c . \square

Proof of Theorem 2.3(3). The existence of transition waves is established in [63]. In the following, we outline the construction of transition waves from [63] and show that they are asymptotically stable by using the arguments in the proof of Theorem 2.2.

Recall that

$$\lambda_0 = \sup \sigma[\partial_{xx}^2 + a(\cdot)], \quad \lambda_1 = 2a_-.$$

Note that $\lambda_0 \geq a_-$. By the arguments of [63, Theorem 1.1], for any $\lambda > \lambda_0$, there is a unique $\phi_\lambda(x)$ such that

$$\phi_\lambda''(x) + a(x)\phi_\lambda(x) = \lambda\phi_\lambda(x) \quad \text{for } x \in \mathbb{R}$$

and

$$\phi_\lambda(0) = 1, \quad \lim_{x \rightarrow \infty} \phi_\lambda(x) = 0.$$

Fix $\lambda \in (\lambda_0, \lambda_1)$ and $1 - \frac{\lambda_1 - \lambda}{a_+} < \alpha < 1$. Let $U_{g, \sqrt{\alpha}}(x)$ be the traveling front profile for the PDE

$$u_t = u_{xx} + g(u)$$

with propagating speed $c_{1, \sqrt{\alpha}} = \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} > 2$ and

$$\lim_{x \rightarrow \infty} U_{g, \sqrt{\alpha}}(x) e^{\sqrt{\alpha} x} = 1.$$

Let

$$h_{g, \alpha}(v) = U_{g, \sqrt{\alpha}}(-\alpha^{-\frac{1}{2}} \ln v)$$

for $v > 0$ and $h_{g, \sqrt{\alpha}}(0) = 0$. Then $h'_{g, \sqrt{\alpha}}(0) = 1$, and by [63, (2.5)],

$$h_{g, \sqrt{\alpha}}(v) \leq v \quad \forall v \in [0, \infty).$$

For any $\lambda \in (\lambda_0, \lambda_1)$, let

$$u^+(t, x) = \phi_\lambda(x) e^{\lambda t} \quad \text{and} \quad u^-(t, x) = h_{g, \sqrt{\alpha}}(\phi_\lambda(x) e^{\lambda t}).$$

By [63, Lemma 2.1, Theorem 1.1], $u^+(t, x)$ is a super-solution of (1.1) and $u^-(t, x)$ is a sub-solution of (1.1), and there is a transition wave $u(t, x) = U_\lambda(t, x)$ of (1.1) satisfying

$$u^-(t, x) \leq U_\lambda(t, x) \leq u^+(t, x). \quad (5.3)$$

We show that this transition wave is asymptotically stable by applying the arguments in the proof of Theorem 2.2.

To this end, first, let

$$w^+(t, x) = u^+(t, x) - u^-(t, x) (\geq 0).$$

We have

$$\begin{aligned} w_t^+(t, x) - w_{xx}^+ - a(x)w^+(t, x) &= -(u_t^- - u_{xx}^- - a(x)u^-(t, x)) \\ &\geq -u^-(t, x)f(x, u^-(t, x)) + a(x)u^-(t, x) \\ &= u^-(t, x)(a(x) - f(x, u^-(t, x))) \\ &\geq 0. \end{aligned}$$

This implies that

$$d_1 w_t^+(t, x) \geq d_1 w_{xx}^+ + d_1 a(x)w^+(t, x)$$

for any $d_1 > 0$. Let

$$\phi(t, x) = u^+(t, x), \quad \phi_1(t, x) = w^+(t, x).$$

We have that $u(t, x) = d\phi(t, x) + d_1\phi_1(t, x)$ is a super-solution of (1.1) for any $d, d_1 > 0$. Note that

$$\lim_{x \rightarrow \infty} \phi(t, x) = \lim_{x \rightarrow \infty} \phi_1(t, x) = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \phi(t, x) = \lim_{x \rightarrow -\infty} \phi_1(t, x) = \infty$$

locally uniformly in t .

Next, for any $M > 0$, let

$$g_M(u) = g(u) - Mu^2$$

and $U_{g_M, \sqrt{\alpha}}$ be the traveling front profile of

$$u_t = u_{xx} + g_M(u)$$

with propagating speed $\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}$ and $\lim_{x \rightarrow \infty} U_{g_M, \sqrt{\alpha}}(x)e^{\sqrt{\alpha}x} = 1$. Let

$$h_{g_M, \sqrt{\alpha}}(v) = U_{g_M, \sqrt{\alpha}}(-\alpha^{-\frac{1}{2}} \ln v)$$

for $v > 0$ and $h_{g_M, \sqrt{\alpha}}(0) = 0$. Then

$$h'_{g_M, \sqrt{\alpha}}(0) = 1 \quad \text{and} \quad h_{g_M, \sqrt{\alpha}}(v) \leq h_{g, \sqrt{\alpha}}(v) \leq v.$$

Similarly, by [63, Lemma 2.1, Theorem 1.1], we have that $\psi_M(t, x) := h_{g_M, \sqrt{\alpha}}(\phi_\lambda(x)e^{\lambda t})$ is a sub-solution of (1.1). This also implies that $u = d\psi_M(t, x)$ is a sub-solution of (1.1) for any $0 < d \leq 1$.

Now, note that for any $M > 0$ and $d_1 > 0$,

$$\psi_M(t, x) \leq U_\lambda(t, x) \leq \phi(t, x) + d_1 \phi_1(t, x) \quad \forall t, x \in \mathbb{R} \quad (5.4)$$

and

$$\lim_{M \rightarrow \infty} \psi_M(t, x) = 0$$

uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}$. Hence for any given u_0 satisfying (2.1) and (2.2), for any $\epsilon > 0$, there are $M > 0$ and $d_1 > 0$ such that

$$(1 - \epsilon)\psi_M(t_0, x) \leq u_0(x) \leq (1 + \epsilon)\phi(t_0, x) + d_1 \phi_1(t_0, x)$$

and then

$$(1 - \epsilon)\psi_M(t, x) \leq u(t, x; t_0, u_0) \leq (1 + \epsilon)\phi(t, x) + d_1 \phi_1(t, x) \quad \forall t \geq t_0, x \in \mathbb{R}. \quad (5.5)$$

By (5.4) and (5.5), we have that

$$u(t, x; t_0, u_0) \leq (1 + \epsilon)U_\lambda(t, x) \frac{\phi(t, x)}{\psi_M(t, x)} \left(1 + d_1 \frac{\phi_1(t, x)}{\phi(t, x)}\right) \quad \forall t \geq t_0, x \in \mathbb{R} \quad (5.6)$$

and

$$u(t, x; t_0, u_0) \geq (1 - \epsilon)U_\lambda(t, x) \frac{\psi_M(t, x)}{\phi(t, x)} \left(1 + d_1 \frac{\phi_1(t, x)}{\phi(t, x)}\right)^{-1} \quad \forall t \geq t_0, x \in \mathbb{R}. \quad (5.7)$$

Let $X(t)$ be an interface location of $U_\lambda(t, x)$. By [24, Proposition 4.2], $X(t)$ satisfies (2.4). Note that

$$\lim_{x \rightarrow \infty} U_\lambda(t, x + X(t)) = 0$$

uniformly in $t \in \mathbb{R}$. By (5.3) and (5.4),

$$\lim_{x \rightarrow \infty} h_{g_M, \sqrt{\alpha}}(\phi_\lambda(x + X(t))e^{\lambda t}) = \lim_{x \rightarrow \infty} h_{g, \alpha}(\phi_\lambda(x + X(t))e^{\lambda t}) = 0$$

uniformly in $t \in \mathbb{R}$. This implies that

$$\lim_{x \rightarrow \infty} \phi_\lambda(x + X(t))e^{\lambda t} = 0$$

uniformly in $t \in \mathbb{R}$. Hence

$$\lim_{x \rightarrow \infty} \frac{\psi_M(t, x + X(t))}{\phi(t, x + X(t))} = \lim_{x \rightarrow \infty} \frac{h_{g_M, \sqrt{\alpha}}(\phi_\lambda(x + X(t))e^{\lambda t})}{\phi_\lambda(x + X(t))e^{\lambda t}} = 1$$

and

$$\lim_{x \rightarrow \infty} \frac{\phi_1(t, x + X(t))}{\phi(t, x + X(t))} = \lim_{x \rightarrow \infty} \left(1 - \frac{h_{g, \sqrt{\alpha}}(\phi_\lambda(x + X(t))e^{\lambda t})}{\phi_\lambda(x + X(t))e^{\lambda t}} \right) = 0$$

uniformly in $t \in \mathbb{R}$. This together with (5.6) and (5.7) implies that, for any given $\epsilon_0 > 0$ with $\frac{\epsilon_0}{4+2\epsilon_0} > \epsilon$, for any $s \geq 0$, there is $x_s(\geq X(t_0 + s))$ such that

$$\sup_{s \geq 0, t \in [t_0 + s, t_0 + s + \tau]} |x_s - X(t)| < \infty \quad (5.8)$$

and

$$\frac{1}{1 + \epsilon_0/2} U_\lambda(t, x) \leq u(t, x; t_0, u_0) \leq (1 + \epsilon_0/2) U_\lambda(t, x) \quad \forall t \in [t_0 + s, t_0 + s + \tau], \quad x \geq x_s. \quad (5.9)$$

It then follows from the arguments after (4.6) in the proof of Theorem 2.2 that, for any $\epsilon > 0$,

$$\rho(u(t + t_0, \cdot; t_0, u_0), U_\lambda(t + t_0, \cdot)) < \epsilon \quad \text{for some } t > 0.$$

Then by Proposition 3.3(1),

$$\rho(u(t + t_0, \cdot; t_0, u_0), U_\lambda(t + t_0, \cdot)) < \epsilon \quad \text{for some } t \gg 1.$$

Therefore, $u = U_\lambda(t, x)$ is asymptotically stable. \square

Proof of Theorem 2.3 (4). Note that the existence of transition waves is established in [48]. In the following, we outline the construction of transition waves from [48] and show that they satisfy the conditions in Theorem 2.2 and hence are asymptotically stable.

First of all, consider

$$v_t = \int_{\mathbb{R}} e^{-\mu(y-x)} \kappa(y-x) v(t, y) dy - v(t, x) + a(t, x) v(t, x), \quad x \in \mathbb{R}, \quad (5.10)$$

where $\mu \in \mathbb{R}$, $a(t, x) = f(t, x, 0)$. Assume that $a(t, x + p) = a(t + T, x) = a(t, x)$. By [48, Propositions 3.2, 3.4, 3.5], we have

(i) For given $\mu > 0$, there are $\lambda(\mu) \in \mathbb{R}$ and a continuous function $v(t, x; \mu)$ such that

$$v(t + T, x; \mu) = v(t, x + p; \mu) = v(t, x; \mu), \quad \inf_{(t, x) \in \mathbb{R} \times \mathbb{R}} v(t, x; \mu) > 0,$$

$$\|v(\cdot, \cdot; \mu)\|_\infty = 1,$$

and

$$v(t, x) := e^{\lambda(\mu)t} v(t, \cdot; \mu)$$

is a solution of (5.10).

(ii) There is $\mu^* > 0$ such that

$$\frac{\lambda(\mu^*)}{\mu^*} = \inf_{\mu > 0} \frac{\lambda(\mu)}{\mu}$$

and

$$\frac{\lambda(\mu)}{\mu} > \frac{\lambda(\mu^*)}{\mu^*} \quad \text{for } 0 < \mu < \mu^*.$$

Let $c^* = \frac{\lambda(\mu^*)}{\mu^*}$. We show that Theorem 2.3 (4) holds with this c^* .

To this end, for given $0 < \mu < \mu^*$, choose μ_1 such that $\mu < \mu_1 < \min\{\mu^*, 2\mu\}$. Let $c_\mu = \frac{\lambda(\mu)}{\mu}$. Let

$$\phi(t, x) = e^{-\mu(x - \frac{1}{\mu}c_\mu t)} v(t, x; \mu), \quad \phi_1(t, x) = e^{-\mu_1(x - \frac{1}{\mu}c_\mu t)} v(t, x; \mu_1),$$

where $v(t, x; \mu)$ and $v(t, x; \mu_1)$ are as in (i). By the arguments of [56, Propositions 3.2, 3.5] and [48, Propositions 5.1, 5.2], there is $d_0 > 0$ such that for any $0 < d < 1$, $d_1 > d_0 d$, and any $t_0 \in \mathbb{R}$, $u_0 \in C_{\text{unif}}^b(\mathbb{R})$ ($u_0(x) \geq 0$) satisfying

$$d\phi(t_0, x) - d_1\phi_1(t_0, x) \leq u_0(x) \leq d\phi(t_0, x) + d_1\phi_1(t_0, x),$$

there holds

$$d\phi(t, x) - d_1\phi_1(t, x) \leq u(t, x; t_0, u_0) \leq d\phi(t, x) + d_1\phi_1(t, x) \quad \forall t \geq t_0. \quad (5.11)$$

Fix $0 < d^* < 1$ and $d_1^* > d_0 d^*$. By the arguments of [56, Theorem 2.4] and [48, Theorem 5.1], there is a uniformly continuous periodic transition wave solution $u = U(t, x)$ satisfying

$$d^*\phi(t, x) - d_1^*\phi_1(t, x) \leq U(t, x) \leq d^*\phi(t, x) + d_1^*\phi_1(t, x). \quad (5.12)$$

Clearly, $X(t) = \frac{c_\mu}{\mu}t$ is an interface location of $U(t, x)$ and satisfies (2.4), and $\phi(t, x)$, $\phi_1(t, x)$ satisfy (2.5) and (2.6). By (5.11), (5.12), and Theorem 2.2, we have that $u = U(t, x)$ is a periodic wave solution with speed $c_\mu = \frac{\lambda(\mu)}{\mu}$ and is asymptotically stable. \square

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